

Exercise Sheet Solutions #6

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- P1.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a random variable, i.e., $\forall A \in \mathcal{B}(\mathbb{R}), X^{-1}(A) \in \mathcal{F}$. Prove that the distribution function of X , $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by $F(x) = \mathbb{P}(X \leq x)$, determines the measure induced by X , $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ defined by $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A))$.

Solution: Notice that $F(x) = \mathbb{P}(X \leq x) = \mathbb{P}_X((-\infty, x])$. In addition, for $y \leq x$, we have $F(x) - F(y) = \mathbb{P}_X((y, x])$. Thus, by this and the continuity of the measure, we conclude that F determines the values of $\mathbb{P}_X$ in the semi-algebra:

$$S = \{(a, b] \mid a, b \in \overline{\mathbb{R}}\}. \quad (1)$$

As this semialgebra generates $\mathcal{B}(\mathbb{R})$, by Caratheodory's theorem - and the fact that $\mathbb{P}_X$ is sigma finite by being a probability measure- we conclude that there is a unique extension to $\mathcal{B}(\mathbb{R})$.

- P2.** Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue measure. Let μ be a measure on $\mathcal{B}(\mathbb{R})$ that satisfies the following conditions:

- i) For all $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$: $\mu(A) = \mu(A + x)$.
- ii) $0 < \mu((0, 1]) < \infty$.

Show that there exists $\alpha > 0$ such that $\mu = \alpha\lambda$.

Solution: Notice that the natural candidate for α is $\mu((0, 1])$.

Let us first prove that for $a, b \in \overline{\mathbb{R}}$, $\mu((a, b]) = \alpha\lambda((a, b])$. Indeed, we start taking $a, b \in \mathbb{Z}$ with $a < b$. In this case, using that $(i, i + 1] - i = (0, 1]$ for $i \in \mathbb{Z}$ and the fact that μ is invariant under translation

$$\begin{aligned} \mu((a, b]) &= \mu\left(\bigsqcup_{i=a}^{b-1} (i, i + 1]\right) \\ &= \sum_{i=a}^{b-1} \mu((i, i + 1]) \\ &= \sum_{i=a}^{b-1} \mu((0, 1]) \\ &= (b - a) \cdot \alpha \\ &= \alpha \cdot \lambda((a, b]). \end{aligned}$$

For extending this for $a, b \in \mathbb{Q}$, we write $a = p_a/q$ and $b = p_b/q$ with $q \in \mathbb{N}$ and $p_a, p_b \in \mathbb{Z}$. Calling $p = p_b - p_a$ and using the translation invariance once again

$$\mu((a, b]) = \mu\left((0, \frac{p_b - p_a}{q}]\right) = \mu\left((0, \frac{p}{q}]\right). \quad (2)$$

On the other hand, we observe that

$$\begin{aligned}
\mu((0, p)) &= \mu\left(\bigsqcup_{i=0}^{q-1} \left(\frac{p}{q}i, \frac{p}{q}(i+1)\right)\right) \\
&= \sum_{i=0}^{q-1} \mu\left(\left(\frac{p}{q}i, \frac{p}{q}(i+1)\right)\right) \\
&= \sum_{i=0}^{q-1} \mu\left(\left(0, \frac{p}{q}\right]\right) \\
&= q \cdot \mu\left(\left(0, \frac{p}{q}\right]\right).
\end{aligned}$$

In consequence

$$\mu((a, b]) = \frac{1}{q} \mu((0, p)) = \frac{p}{q} \cdot \mu((0, 1]) = \alpha \cdot \lambda((a, b]). \quad (3)$$

By continuity of the measures, we get that

$$\mu((a, b]) = \alpha \lambda((a, b]) \quad (4)$$

for each $a, b \in \overline{\mathbb{R}}$. Hence, we conclude that the measure μ and $\alpha\lambda$ coincide in the generating semialgebra

$$\mathcal{S} = \{(a, b] \mid a, b \in \overline{\mathbb{R}}\}, \quad (5)$$

and by Carathéodory extension theorem, we conclude that $\mu = \alpha\lambda$ by the fact that both measures are sigma finite.

P3. In this exercise, we prove that there exists a Lebesgue non-measurable subset of $\mathbb{R}$. For this, we define an equivalence relation on $[0, 1]$ by $x \mathcal{R} y$ if $y - x \in \mathbb{Q}$. By axiom of choice, let $E \subseteq [0, 1]$ be a set containing exactly one representative of each equivalence class, and for each $t \in \mathbb{Q} \cap [0, 1]$ let $E_t = \{x + t \bmod 1 \mid x \in E\} \subseteq [0, 1]$.

- (a) Show that the sets $(E_t)_{t \in \mathbb{Q} \cap [0, 1]}$ are pairwise disjoint.
- (b) Show that

$$\bigsqcup_{t \in \mathbb{Q} \cap [0, 1]} E_t = [0, 1].$$

- (c) Assume by contradiction that E is Lebesgue measurable. Show that for every $t \in \mathbb{Q} \cap [0, 1]$, E_t is Lebesgue measurable and $\lambda(E_t) = \lambda(E)$.
- (d) Conclude by arriving to a contradiction.

Solution: See Theorem 5.29 in the lecture notes.

P4. Consider λ as the Lebesgue measure on $\mathbb{R}$, and let A be a Lebesgue measurable set. Prove that if $\lambda(A) > 0$, then A contains a non-measurable set.

Solution: Let $V \subseteq [0, 1]$ be a Vitali's set (this is a set E as in the proof of theorem 5.29 in the lecture notes). We first prove that every measurable subset C of V must have measure 0. Indeed, notice that $\{C + q\}_{q \in \mathbb{Q} \cap [0, 1]}$ are all disjoint and measurable. Hence

$$\sum_{q \in \mathbb{Q} \cap [0, 1]} \lambda(C + q) = \lambda\left(\bigcup_{q \in \mathbb{Q} \cap [0, 1]} C + q\right) \leq \lambda([-1, 2]) = 3,$$

as all terms in the sum are equal, it is imperative that $\lambda(C) = 0$.

Now, let A be a measurable set with $\lambda(A) > 0$. Without loss of generality, we can assume $A \subseteq [0, 1]$, because we can always find $n \in \mathbb{Z}$ with $\mu(A \cap [n, n+1]) > 0$, and then $(A - n) \cap [0, 1]$ must have positive measure, so by replacing A by $A - n$ we can assume this. Notice that

$$A = \bigsqcup_{q \in \mathbb{Q} \cap [0, 1]} A \cap (V + q), \quad (6)$$

and if there is $q \in \mathbb{Q} \cap [0, 1]$ such that $A \cap (V + q)$ is not measurable, we are done. In consequence, we can assume by contradiction that $A \cap (V + q)$ are all measurable, and then they must have measure 0, given that $A \cap (V + q) = (A - q) \cap V$. Consequently

$$0 \leq \lambda(A) = \lambda\left(\bigsqcup_{q \in \mathbb{Q} \cap [0, 1]} A \cap (V + q)\right) \leq \sum_{q \in \mathbb{Q} \cap [0, 1]} \lambda(A \cap (V + q)) = 0,$$

which is a contradiction.